Numerik 2 – Übung02 – Georg Kuschk

1.a i)

$$c_{(A,B)}(l) = \det(A - lB) = \det\begin{pmatrix} -1 & -l \\ -l & 1 \end{pmatrix} = -1 - l^2$$

$l^2 + 1 = 0 \quad \Leftrightarrow \quad l_{1,2} = \pm i$ d.h. $s(A,B) = \{-i, +i\}$

1.a ii)

$$c_{(A,B)}(l) = \det(A - lB) = \det\begin{pmatrix} -1 & 0 \\ 0 & -l \end{pmatrix} = l$$

$l = 0$ d.h. $s(A,B) = \{0\}$

1.a iii)

$$c_{(A,B)}(l) = \det(A - lB) = \det\begin{pmatrix} 1-l & 2 \\ 0 & 0 \end{pmatrix} = 0$$

$0 = 0 \quad \Leftrightarrow \quad l \in C$ d.h. $s(A,B) = C$

1.b)

$$Ax = lBx$$

$$(A - lB)x = 0$$

$$\begin{pmatrix} -1 & -l \\ -l & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$\Rightarrow$

I $\quad x_1 = -l x_2$

II $\quad x_2 = l x_1$

Eigenwerte nochmal ausrechnen (siehe 1.a i)) :

II in I : $x_1 = -l^2 x_1$, d.h. $x_1(1 + l^2) = 0 \quad \Rightarrow l^2 = -1 \quad \Rightarrow l_1 = +i \quad l_2 = -i$

zugehörige Eigenvektoren :

$$l_1 = +i \qquad \Rightarrow v_1 = \begin{pmatrix} a \\ a \cdot i \end{pmatrix}, a \in C$$

$$l_2 = -i \qquad \Rightarrow v_2 = \begin{pmatrix} a \\ -a \cdot i \end{pmatrix}, a \in C$$

1.c)

$$Ax = lBx \qquad \Rightarrow \qquad \frac{1}{l} Ax = Bx$$

Für $l \to 0$ gilt $\frac{1}{l} \to \infty$

Desweiteren ist hier $s(A,B) = s(B,A)$

$$\frac{1}{l} \in s(B,A) \Rightarrow \infty \in s(A,B)$$

2.)

$$A_1 = \begin{pmatrix} 5 & 0 & 0 & 1 \\ 0 & 5 & 1 & 1 \\ 0 & 1 & 5 & 0 \\ 1 & 1 & 0 & 5 \end{pmatrix}$$ Transformation auf Hessenberg-Form

$$a_1 = \begin{pmatrix} a_{2,1} \\ a_{3,1} \\ a_{4,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \alpha_1 = \|a_1\|_2 = 1 \quad \Rightarrow \tilde{v}_1 = a_1 \pm \alpha_1 e_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$v_1 = \begin{pmatrix} 0 \\ \tilde{v}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$Q_1 = I - 2\frac{v_1 v_1^T}{v_1^T v_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$Q_1 A_1 = \begin{pmatrix} 5 & 0 & 0 & 1 \\ -1 & -1 & 0 & -5 \\ 0 & 1 & 5 & 0 \\ 0 & -5 & -1 & -1 \end{pmatrix} \quad , \quad A_2 = Q_1 A_1 Q_1^T = \begin{pmatrix} 5 & -1 & 0 & 0 \\ -1 & 5 & 0 & 1 \\ 0 & 0 & 5 & -1 \\ 0 & 1 & -1 & 5 \end{pmatrix}$$

$$a_1 = \begin{pmatrix} a_{3,2} \\ a_{4,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad \alpha_1 = \|a_1\|_2 = 1 \quad \Rightarrow \tilde{v}_2 = a_1 \pm \alpha_1 e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$v_2 = \begin{pmatrix} 0 \\ \tilde{v}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$Q_2 = I - 2\frac{v_2 v_2^T}{v_2^T v_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$Q_2 A_2 = \begin{pmatrix} 5 & -1 & 0 & 0 \\ -1 & 5 & 0 & 1 \\ 0 & -1 & 1 & -5 \\ 0 & 0 & -5 & 1 \end{pmatrix} \quad , \quad A_3 = Q_2 A_2 Q_2^T = \begin{pmatrix} 5 & -1 & 0 & 0 \\ -1 & 5 & -1 & 0 \\ 0 & -1 & 5 & -1 \\ 0 & 0 & -1 & 5 \end{pmatrix}$$

und fertig

3.) $A = \begin{pmatrix} 2 & e \\ e & 1 \end{pmatrix}$, $|e| << 1$

$$A - s_1 I = QR \qquad R = Q^T (A - s_1 I)$$
$$A_1 = RQ + s_1 I$$

$$R = \begin{pmatrix} \cos j & \sin j \\ -\sin j & \cos j \end{pmatrix} \cdot \left(A - s_1 I\right) = \begin{pmatrix} \cos j & \sin j \\ -\sin j & \cos j \end{pmatrix} \cdot \begin{pmatrix} 2 - s_1 & e \\ e & 1 - s_1 \end{pmatrix}$$

$$r_{21} \overset{!}{=} 0$$
$$0 = -(\sin j)(2 - s_1) + e \cdot \cos j$$

$$\tan j = \frac{\sin j}{\cos j} = \frac{e}{2 - s_1}$$
$$\sin j = \frac{\tan j}{\sqrt{1 + \tan^2 j}} = \frac{e}{\sqrt{(2 - s_1)^2 + e^2}}$$
$$\cos j = \frac{1}{\sqrt{1 + \tan^2 j}} = \frac{2 - s_1}{\sqrt{(2 - s_1)^2 + e^2}}$$

$$\Rightarrow R = \cos j \cdot \begin{pmatrix} \dfrac{e}{\tan j} + e \cdot \tan j & 2e - \tan j \\ 0 & \dfrac{e}{\tan j} - e \cdot \tan j - 1 \end{pmatrix}$$
$$\Rightarrow RQ = \cos^2 j \cdot \begin{pmatrix} * & e - e \cdot \tan^2 j - \tan j \\ e - e \cdot \tan^2 j - \tan j & * \end{pmatrix}$$

a) $s_1 = 0 \quad \Rightarrow \quad \tan j = \dfrac{e}{2}$

$$\Rightarrow A_1 = \cos^2 j \cdot \begin{pmatrix} * & \dfrac{1}{2}\left(e - \dfrac{e^3}{2}\right) \\ \dfrac{1}{2}\left(e - \dfrac{e^3}{2}\right) & * \end{pmatrix} \approx \begin{pmatrix} * & \dfrac{e}{2} \\ \dfrac{e}{2} & * \end{pmatrix}$$

$$\cos^2 j = \frac{4}{4 + e^2} \approx 1$$

b) $s_1 = 1 \quad \Rightarrow \quad \tan j = e$

$$\Rightarrow A_1 = \cos^2 j \cdot \begin{pmatrix} * & -e^3 \\ -e^3 & * \end{pmatrix}$$

$$\cos^2 j = \frac{1}{1 + e^2} \approx 1$$