Numerik 2 – Übung05 – Georg Kuschk

1.a)

$$J_0(t) = \frac{1}{p}\int_0^p \cos(t\cdot\sin x)dx$$

$$\begin{aligned}
J_0^{(k)}(t) &= \frac{1}{p}\int_0^p \frac{d^k}{dt^k}\cos(t\cdot\sin x)dx \\
&= \frac{1}{p}\int_0^p (-1)^{\left\lfloor \frac{k}{2} \right\rfloor}\cdot(\sin x)^k\cdot\begin{cases}\cos(t\cdot\sin x) & \text{für } k \text{ gerade} \\ \sin(t\cdot\sin x) & \text{für } k \text{ ungerade}\end{cases} dx
\end{aligned}$$

$$\begin{aligned}
\left|J_0 - I\cdot J_0\right| &\le \frac{\left|J_0''(x)\right|}{2!}(t-t_i)(t_{i+1}-t) \qquad \text{für } t\in[t_i,t_{i+1}] \text{ mit } t_{i+1}-t_i = h \\
&\le \max_{t\in R^+}\frac{\left|J_0''(t)\right|}{2}\cdot\frac{h^2}{4} \\
&\le \frac{h^2}{8}\cdot\max\left|\frac{1}{p}\int_0^p (\sin^2 x)\cos(t\cdot\sin x)\ dx\right| \\
&\le \frac{h^2}{8}\cdot\max\left|\frac{1}{p}\underbrace{\int_0^p (\sin^2 x)\ dx}_{=p/2}\right| \\
&\le \frac{h^2}{16}
\end{aligned}$$

$$\frac{h^2}{16}\le 10^{-6} \quad\Leftrightarrow\quad h\le 4\cdot 10^{-3}$$

1.b)

$$\begin{aligned}
\left|P_n(t) - J_0\right| &= \frac{1}{(n+1)!}\cdot\underbrace{w(t)}_{|\cdot|\le 1}\cdot\underbrace{J_0^{(n+1)}(x)}_{|\cdot|\le 1} \\
&\le \frac{1}{(n+1)!}\xrightarrow[n\to\infty]{} 0
\end{aligned}$$

Fehler bei kubischer Spline-Interpolation auf $[0,1]$ mit Stützstellen t_i aus Aufgabe b)

$$\left|S_{4,\Delta_n}(t) - J_0(t)\right| \le 2\cdot\max_{t\in[0,1]}\left|J_0^{(4)}(t)\right|\cdot h^4 \le \frac{2}{n^4}$$

2.)

$$g(y) = \cos y \cdot \cosh(y) + 1$$

$$x_0 = g(y_0) \approx 0.294 \quad \Rightarrow y_0 = 1.8$$

$$x_1 = g(y_1) \approx -0.105 \quad \Rightarrow y_1 = 1.9$$

$$\Rightarrow y_{01} = \frac{y_1 - y_0}{x_1 - x_0} = -0.25$$

$$y_2 = P_1(0) = y_0 + y_{01} \cdot (0 - x_0) \approx 1.8737$$

$$x_2 = g(y_2) \approx 5.8 \cdot 10^{-3} \qquad \Rightarrow y_2 = 1.8737$$

$$\Rightarrow y_{12} = -2.37 \cdot 10^{-1}$$

$$\Rightarrow y_{012} = -4.57 \cdot 10^{-2}$$

$$y_3 = P_2(0) = P_1(0) + y_{012} \cdot (0 - x_0)(0 - x_1) \approx 1.8751$$

$$x_3 = g(y_3) \approx -10^{-5} \quad \Rightarrow y_3 = 1.8751$$

$$\Rightarrow y_{13} = -2.4 \cdot 10^{-1}$$

$$\Rightarrow y_{123} = -4.17 \cdot 10^{-2}$$

$$\Rightarrow y_{0123} \approx -1.365 \cdot 10^{-2}$$

$$y_4 = P_3(0) = P_2(0) + y_{0123} \cdot (0 - x_0)(0 - x_1)(0 - x_2) \approx 1.8751$$

$$x_4 = g(y_4) \approx -3 \cdot 10^{-11}$$

3.)

$$b(t_0,t_1) = \max_{t\in I}\left|(t-t_0)(t-t_1)\right|$$
$$= \max_{t\in I}\left|t^2-(t_0+t_1)\cdot t+t_0t_1\right|$$

max in $(-1,1)$: $t_m = \frac{1}{2}(t_0+t_1)$

$$b(t_0,t_1) = \max\left\{\underbrace{\frac{1}{4}(t_1-t_0)^2}_{b_1} \quad , \quad \underbrace{\left|(-1-t_0)(-1-t_1)\right|}_{b_2} \quad , \quad \underbrace{\left|(1-t_0)(1-t_1)\right|}_{b_3}\right\}$$

b ist minimal für $b_1 = b_2 = b_3$

$b_2 = b_3$:

$$\Rightarrow (-1-t_0)(-1-t_1) = (1-t_0)(1-t_1)$$
$$\Rightarrow t_0+t_1 = 0$$
$$\Rightarrow t_0 = -t_1$$

$b_1 = b_3$:

$$\Rightarrow \frac{1}{4}(t_1+t_1)^2 = (1+t_1)(1-t_1)$$
$$\Rightarrow t_1^{\,2} = 1-t_1^{\,2}$$
$$\Rightarrow t_1^{\,2} = \frac{1}{2}$$
$$\Rightarrow t_{0,1} = \pm\frac{\sqrt{2}}{2}$$

Zusammenhang :

$$\left(t-\frac{\sqrt{2}}{2}\right)\left(t+\frac{\sqrt{2}}{2}\right) = t^2-\frac{1}{2} \quad = \frac{1}{2}T_2(t) \qquad \text{(Tschebyscheffpolynom)}$$
$$= \frac{1}{2}\cos(2\arccos t)$$