Numerik 2 – Übung06 – Georg Kuschk

1.a)

$$s_f(x_i) = f(x_i)$$

$$s_f(x) = a_0 + b_0(x - x_i) \qquad \text{für} \quad x \in [x_i\ ,\ x_{i+1}]$$

$$\Rightarrow a_0 = f(x_i)$$

$$\Rightarrow b_0 = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

$$\Rightarrow s_f(x) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i)$$

1.b)

$$\|g'\|_2^2 = \int_a^b (g')^2\,dx = \int_a^b \left(s_f' + (g' - s_f')\right)^2 dx$$

$$= \int_a^b (s_f')^2\,dx \quad + \quad \underbrace{2\int_a^b s_f'(g' - s_f')dx}_{\text{zu zeigen:}\ =0} \quad + \quad \underbrace{\int_a^b (g' - s_f')^2\,dx}_{\geq 0}$$

$$\geq \int_a^b (s_f')^2\,dx \quad = \quad \|s_f'\|_2^2$$

zu zeigen :

$$\int_a^b s_f'(g' - s_f')dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} s_f'(g' - s_f')dx = \sum_{i=0}^{n-1}\left(\left[s_f'(g - s_f)\right]_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} s_f''(g - s_f)dx\right)$$

$$= \sum_{i=0}^{n-1}\left(s_f'(x_{i+1}) \cdot \underbrace{\left(g(x_{i+1}) - s_f(x_{i+1})\right)}_{=0} \quad - \quad s_f'(x_i) \cdot \underbrace{\left(g(x_i) - s_f(x_i)\right)}_{=0} \quad - \quad \int_{x_i}^{x_{i+1}} \underbrace{s_f''}_{=0}(g - s_f)dx\right)$$

$$= 0$$

$$\Rightarrow \quad \|g'\|_2 \geq \|s_f'\|_2$$

1.c)

$$\|g'\|_2^2 \geq \|s_f'\|_2^2 = \int_a^b (s_f')^2\,dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} (s_f')^2\,dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\left(\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}\right)^2 dx$$

$$= \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \frac{\left(f(x_{i+1}) - f(x_i)\right)^2}{x_{i+1} - x_i}\,dx = \sum_{i=0}^{n-1}\frac{\left(g(x_{i+1}) - g(x_i)\right)^2}{x_{i+1} - x_i}$$

2.)

$\underline{i=1}$:

$$\frac{h_0}{6}c_0+\frac{h_0+h_1}{3}c_1+\frac{h_1}{6}c_2 = f \ldots.$$

$$c_0 = S_\Delta''(a) = S_\Delta''(b) = c_{l+1}$$

$\underline{i=l+1}$:

$$\frac{h_l}{6}c_l+\frac{h_l+h_{l+1}}{3}c_{l+1}+\frac{h_{l+1}}{6}c_{l+2} = f \ldots.$$

$c_{l+2} = c_1$
$h_{l+1} = h_0$
$f_{l+2} = f_1$

$\Rightarrow$

$$\begin{pmatrix} \frac{h_0+h_1}{3} & \frac{h_1}{6} & 0 & \cdots & \frac{h_0}{6} \\ \frac{h_1}{6} & \frac{h_1+h_2}{3} & \frac{h_2}{6} & & \\ 0 & \ddots & \ddots & \ddots & \\ \vdots & & \frac{h_{l-1}}{6} & \frac{h_{l-1}+h_l}{3} & \frac{h_l}{6} \\ \frac{h_0}{6} & & & \frac{h_l}{6} & \frac{h_l+h_0}{3} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{l+1} \end{pmatrix} = \begin{pmatrix} \frac{f_2-f_1}{h_1}-\frac{f_1-f_0}{h_0} \\ \vdots \\ \vdots \\ \frac{f_1-f_0}{h_0}-\frac{f_{l+1}-f_l}{h_l} \end{pmatrix}$$

$\Rightarrow$

$$c_0 = \frac{23}{4} \quad , \quad c_1 = -\frac{7}{4} \quad , \quad c_2 = \frac{7}{8} \quad , \quad c_3 = -\frac{13}{4}$$

$i=0$: $[0,1]$: $s_0(t) = -\frac{15}{24}t+\frac{23}{8}t^2-\frac{15}{12}t^3$

$i=1$: $[1,3]$: $s_1(t) = 1+\frac{11}{8}(t-1)-\frac{7}{8}(t-1)^2+\frac{11}{6\cdot 16}(t-1)^3$

$i=2$: $[3,5]$: $s_2(t) = 2+\frac{1}{2}(t-3)+\frac{7}{16}(t-3)^2-\frac{33}{6\cdot 16}(t-3)^3$

$i=3$: $[5,6]$: $s_3(t) = 2-\frac{15}{8}(t-5)+\frac{13}{8}(t-5)^2+\frac{5}{6}(t-5)^3$

3.a)

Randbedingung : $f_0 = f_n = 0$

$\Rightarrow s'$ hat eine Nullstelle in (t_0, t_n) - (Satz von Rolle)

Randbedingung : $s'(t_0) = 0$, $s'(t_n) = 0$

$\Rightarrow s''$ hat zwei getrennte Nullstellen in (t_0, t_n)

s'' ist stückweise linear durch $(t_0, 0)$, (t_1, c_1) , ... , (t_l, c_l) , $(t_{l+1}, 0)$

$\Rightarrow c_1 = 0$, $c_2 = 0$ für $l = 2$

und $c_1 = 0$ für $l = 1$

3.b)

$\underline{t = 0 \quad , \quad l = 3 \quad :}$

$s(t_0) = s(t_2) = s(t_4)$

$\Rightarrow s'$ hat Nullstellen t_0, t_4 in (t_0, t_2) und (t_2, t_4)

$\Rightarrow s''$ hat 5 getrennte Nullstellen , stückweise linear $\Rightarrow s'' \equiv 0 \Rightarrow s \equiv 0$

3.c)

$t_i := -2 , -1 , 0 , 1 , 2$

$f := 1$

linke Randbedingung :

$a_0 = b_0 = c_0 = 0$, d_0 frei

$\Rightarrow s(t_1) = \frac{1}{6} d_0$, $s'(t_1) = \frac{1}{2} d_0$, $s''(t_1) = d_0$

rechte Randbedingung :

d_3 frei

$c_3 = -d_3$

$b_3 = \frac{1}{2} d_3$

$a_3 = -\frac{1}{6} d_3$

$f = 2 \Rightarrow a_2 = 2$

Stetigkeitsbedingungen in $t=0$:

$$a_1 + b_1 + \frac{1}{2}c_1 + \frac{1}{6}d_1 = a_2$$

$$b_1 + c_1 + \frac{1}{2}d_1 = b_2$$

$$c_1 + d_1 = c_2$$

Stetigkeitsbedingungen in $t=1$:

$$a_2 + b_2 + \frac{1}{2}c_2 + \frac{1}{6}d_2 = a_3$$

$$b_2 + c_2 + \frac{1}{2}d_2 = b_3$$

$$c_2 + d_2 = d_3$$

$\Rightarrow$ Unbekannte $d_0, d_1, d_2, d_3, b_2, c_2$ und 6 Gleichungen

$\Rightarrow$

$$d_0 = \frac{3}{2} \quad , \quad d_1 = -\frac{9}{2} \quad , \quad d_2 = \frac{9}{2} \quad , \quad d_3 = -\frac{3}{2}$$

$$a_1 = \frac{1}{4} \quad , \quad a_3 = \frac{1}{4} \quad , \quad b_1 = \frac{3}{4} \quad , \quad b_2 = 0 \quad , \quad b_3 = -\frac{3}{4}$$

$$c_1 = \frac{3}{2} \quad , \quad c_2 = -3 \quad , \quad c_3 = \frac{3}{2}$$