Numerik 2 – Übung11 – Georg Kuschk

1.)

Mit $g(x) = x^3$ gilt :

$$\int_a^{a+1} g(x)\,dx = \frac{g(a)+g(a+1)}{2} + \frac{B_2}{2!}\cdot\left(g'(a) - g'(a+1)\right) - 0$$

$$= \frac{a^3 + (a+1)^3}{2} + \frac{1}{12}\left(3a^2 - 3(a+1)^2\right)$$

(Euler-MacLaurinsche Summenformel $, m = 1$)

$$\int_0^n x^3 dx = \sum_{k=0}^{n-1} \int_k^{k+1} x^3 dx$$

$$= \sum_{k=0}^{n-1}\left(\frac{k^3 + (k+1)^3}{2} + \frac{1}{2}\left(3k^2 - 3(k+1)^2\right)\right)$$

$$= \sum_{k=0}^{n-1}\left(\frac{k^3}{2} + \frac{k^2}{4}\right) + \sum_{k=1}^{n}\left(\frac{k^3}{2} - \frac{k^2}{4}\right)$$

$$= \sum_{k=0}^{0}\left(\frac{k^3}{2} + \frac{k^2}{4}\right) + \sum_{k=1}^{n-1}\left(\frac{k^3}{2} + \frac{k^2}{4} + \frac{k^3}{2} - \frac{k^2}{4}\right) + \sum_{k=n}^{n}\left(\frac{k^3}{2} - \frac{k^2}{4}\right)$$

$$\frac{n^4}{4} = \underbrace{0}_{k=0} + \underbrace{\sum_{k=1}^{n-1} k^3}_{k=1,\dots,n-1} + \underbrace{\frac{n^3}{2} - \frac{n^2}{4}}_{k=n}$$

$$\frac{n^4}{4} = \sum_{k=1}^{n} k^3 - n^3 + \frac{n^3}{2} - \frac{n^2}{4}$$

$$\frac{n^4}{4} = \sum_{k=1}^{n} k^3 - \frac{n^3}{2} - \frac{n^2}{4}$$

$$\Rightarrow \quad \sum_{k=1}^{n} k^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

$$= \frac{n^2}{4}\cdot(n^2 + 2n + 1)$$

$$= \left(\frac{n}{2}\cdot(n+1)\right)^2$$

$$= \left(\frac{n(n+1)}{2}\right)^2$$

q.e.d.

2.)

$$p_k(t) := \frac{k!}{(2k)!} \cdot \frac{d^k}{dt^k}\left(t^2 - 1\right)^k$$

2.a)

$$\begin{aligned} p_k(t) &= \frac{k!}{(2k)!} \cdot \frac{d^k}{dt^k}\left(t^{2k} + \ldots\right) \\ &= \frac{k!}{(2k)!} \cdot \frac{d^{k-1}}{dt^{k-1}} \cdot 2k \cdot \left(t^{2k-1} + \ldots\right) \\ &= \frac{k!}{(2k)!} \cdot \left((2k) \cdot (2k-1) \cdot \ldots \cdot (k+1)\right) \cdot \left(t^k + \ldots\right) \\ &= t^k + \ldots \end{aligned}$$

2.b)

$$\begin{aligned} < p_i, p_j > &= \int_{-1}^{1} p_i(t) \cdot p_j(t)\, dt \\ &= c_{ij} \cdot \int_{-1}^{1} \frac{d^i}{dt^i}\left(t^2 - 1\right)^i \cdot \frac{d^j}{dt^j}\left(t^2 - 1\right)^j \, dt \\ &= c_{ij} \cdot \left(\left[\frac{d^i}{dt^i}\left(t^2 - 1\right)^i \cdot \frac{d^{j-1}}{dt^{j-1}}\left(t^2 - 1\right)^j \right]_{-1}^{1} - \int \frac{d^{i+1}}{dt^{i+1}}\left(t^2 - 1\right)^i \cdot \frac{d^{j-1}}{dt^{j-1}}\left(t^2 - 1\right)^j \right) \\ &= -c_{ij} \cdot \int_{-1}^{1} \frac{d^{i+1}}{dt^{i+1}}\left(t^2 - 1\right)^i \cdot \frac{d^{j-1}}{dt^{j-1}}\left(t^2 - 1\right)^j \, dt \\ &= (-1)^{i+1} \cdot c_{ij} \cdot \int_{-1}^{1} \underbrace{\frac{d^{2i+1}}{dt^{2i+1}}\left(t^2 - 1\right)^i}_{\equiv 0} \cdot \frac{d^{j-i-1}}{dt^{j-i-1}}\left(t^2 - 1\right)^j \, dt \\ &= 0 \qquad \text{, für } i < j \end{aligned}$$

3.)

$$I_n = \int_1^2 (\ln t)^n \, dt$$

3.a)

$$\begin{aligned} I_n &= \int_1^2 1 \cdot (\ln t)^n \, dt \\ &= \left[t \cdot (\ln t)^n\right]_1^2 - \int_1^2 t \cdot n \cdot (\ln t)^{n-1} \cdot \frac{1}{e} \, dt \\ &= 2(\ln 2)^n - n \cdot \int_1^2 (\ln t)^{n-1} \, dt \\ &= 2(\ln 2)^n - n \cdot I_{n-1} \end{aligned}$$

3.b)

i.)

$$\hat{I}_1 = I_1 + e_1$$

$$\hat{I}_2 = (2\ln 2)^2 - 2\hat{I}_1$$
$$= (2\ln 2)^2 - 2I_1 - 2e_1$$

...

$$\hat{I}_7 = \underbrace{I_7}_{\approx 10^{-1}} + \underbrace{7! \cdot e_1^*}_{\approx 10^{-1}..10^{-2}}$$

$$\Rightarrow |I_7| \approx |7! \cdot e_1^*|$$ - unbrauchbar

ii.)

$$\hat{I}_7 = I_7 + e_2$$

... wie oben

$$\hat{I}_1 = I_1 + \underbrace{\frac{1}{7!} \cdot e_2^*}_{\approx 10^{-8}}$$

3.c)

$$\hat{I}_{n+k} = I_{n+k} + c$$

$$\hat{I}_{n+k-l} = I_{n+k-l} - (-1)^l \frac{(n+k-l)!}{(n+k)!} \cdot c$$

Für $n = 7$, $k = l$ folgt für den Fehler : $\frac{7!}{(7+k)!} \cdot c \quad \begin{matrix} \le 5 \cdot 10^{-5} \\ \le 5 \cdot 10^{-9} \end{matrix}$

grob : $|I_n| \le 1$ $\Rightarrow c \le 1$ $\Rightarrow k = 5$ bzw. $k = 8$

genauer : $|I_n| \le 0.7^n$ $\Rightarrow |c| \le 0.7^{n+k}$ $\Rightarrow k = 3$ bzw. $k = 6$